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Symmetry breaking and bifurcating solutions in the classical complex ϕ^6 field theory

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Abstract. In this paper we consider the equation of motion for a complex classical field in a $(3+1)$-dimensional ϕ^6 model. The resultant complex non-linear Klein-Gordon equation is solved using an ansatz in which the envelope satisfies a scalar non-linear Klein-Gordon equation and the carrier satisfies either a wave equation or Laplace's equation with additional constraints. We use the results of the symmetry reduction method to exactly solve both the carrier equations and the envelope equation. The latter has recently been analysed and here we only briefly discuss the types of the appropriate solutions. However, we present a detailed discussion on one particular solution which is physically important. It bifurcates both in real space and in phase space. Possible physical applications have been outlined in the last section and they include superfluidity, superconductivity, liquid crystals and helicoidal metamagnets.

1. Introduction

The objective of this paper is to use the method of symmetry reduction for partial differential equations in order to find exact solutions of the equation of motion for the complex order-parameter field ϕ that corresponds to the Lagrangian of equation (1) given below. We intend to present and analyse certain classes of exact solutions which are invariant with respect to subgroups of the symmetry group of the equation of motion. In particular, we shall discuss in detail a bifurcating solution which reflects the phenomenon of symmetry breaking taking place in the system. The aim of this paper is not a complete presentation of all solutions obtained using this method (this will be done elsewhere) but an analysis of some special, physically interesting solutions.

Consider the following classical Lagrangian density:

$$\mathscr{L}=\tfrac{1}{2}m\phi_t\phi_t^*-\tfrac{1}{2}D(\nabla\phi)(\nabla\phi^*)-A_2\phi\phi^*-A_4(\phi\phi^*)^2-A_6(\phi\phi^*)^3 \qquad (1)$$

where ϕ is a complex order-parameter field and $\phi=\phi(t,\boldsymbol{x})$. This type of Lagrangian density leads to the Hamiltonian

$$H=\int \mathrm{d}^N x(\tfrac{1}{2}m|\phi_t|^2+\tfrac{1}{2}D|\nabla\phi|^2+A_2|\phi|^2+A_4|\phi|^2+A_6|\phi|^6) \qquad (2)$$

which is typical of Landau-Ginzburg models of critical phenomena (Landau and Lifshitz 1980, Amit 1978) with complex (two-component) order parameters, as, e.g., in superconductors, superfluids or metamagnets. Following Landau, A_2 changes sign according to $A_2 = a(T - T_c)$. If $A_4 > 0$, the resulting phase transition is of second order and takes place at T_c. If $A_4 < 0$, the transition is of first order and takes place at $T_c^* \equiv T_c + A_4^2/4aA_6$. In both cases $A_6 > 0$. For $A_4 < 0$ the coexistence range is $T_c \leqslant T \leqslant T_0^* \equiv T_c + A_4^2/3aA_6$.

As shown in figure 1, this Lagrangian's stable state may either be non-degenerate (a) or doubly degenerate (b) or triply degenerate (c). Intermediate situations are also possible with metastable states lying above the ground state (d and e) which may be either degenerate or not. Figure 1 just shows the projection of $V \equiv A_2\phi\phi^* + A_4(\phi\phi^*)^2 + A_3(\phi\phi^*)^3$ on the plane of $\phi = \phi^*$. The various potential forms of figure 1 correspond to different conditions imposed on A_2 and A_4 (see figure 2). On the left-hand side of figure 2 we have divided the neighbourhood of the origin of the A_2A_4-phase diagram into regions corresponding to the potential forms of figure 1. The origin of the diagram represents a tricritical point $A_2 = A_4 = 0$ (Aharony 1983) and it can be approached from each of the five regions in a different way. At the

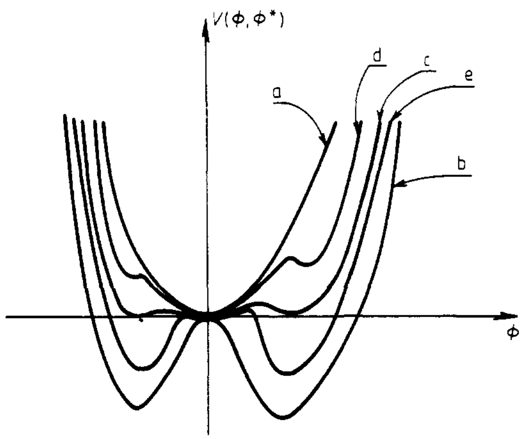

Figure 1. The five possible forms of the nonlinear potential $V(\phi, \phi^*)$ when $\phi = \phi^*$ and $A_4 < 0$. a, $T > T_0^*$; b, $T < T_c$; c, $T = T_c^*$; d, $T_c^* < T < T_0^*$; e, $T_c < T < T_c^*$. If $A_4 > 0$, then a takes place when $T > T_c$ and b when $T \leqslant T_c$.

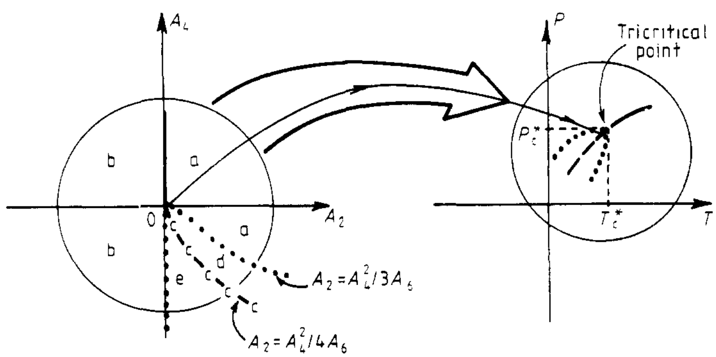

Figure 2. A continuous mapping of the neighbourhood of the origin on the A_2A_4 phase diagram onto the neighbourhood of the tricritical point on the PT phase diagram. - - -, line of first-order phase transitions; ——, line of second-order phase transitions, · · · · · ·, boundary of the coexistence region.

tricritical point an infinitesimal perturbation may destabilise the system and cause it to fall into either one of the five regions.

On crossing from region a to b the system undergoes a second-order phase transition whose line is given by $A_2 = 0$ and $A_4 > 0$. On crossing from region d to e the system undergoes a first-order transition whose line is the half-parabola $A_2 = A_4^2/4A_6$; $A_4 < 0$. The coexistence region is bounded by the half-parabola $A_2 = A_4^2/3A_6$; $A_4 < 0$ and the half-line $A_2 = 0$; $A_4 < 0$. The thermal hysteresis present here means that the metastable disordered phase exists up to the negative-A_4 half-axis on going in a clockwise direction while the metastable ordered phase exists up to the $A_2 = A_4^2/3aA_6$ parabola on going in the anticlockwise direction. Since A_2 depends only on the temperature T (provided T is sufficiently close to T_c) and A_4 depends only on the thermodynamic force (e.g., pressure P or magnetic field), then this generic situation represented on the A_2A_4 diagram for each particular physical system can be uniquely mapped onto the PT phase diagram. Although $A_4 = A_4(P)$ in a way characteristic of the given system, it is always sufficiently smooth close to the tricritical point (T_c^*, P_c^*) for a continuous map from (A_2, A_4) to (T, P) to be isomorphic in the neighbourhood of the tricritical point.

A transformation which leaves the Lagrangian invariant but does not preserve its solution is called spontaneous symmetry breaking. The Lagrangian $\mathscr{L}$ of equation (1) is invariant with respect to the sign reversal of the fields ϕ and ϕ^* (which may result from, e.g., parity or time reversal). The associated stable homogeneous solution ϕ experiences a bifurcation as is shown in figure 3 (with $\phi = \phi^*$) in two distinct cases: $A_4 > 0$ (figure 3(a)) and $A_4 < 0$ (figure 3(b)). In figure 3, ϕ_+ and ϕ_- denote the two

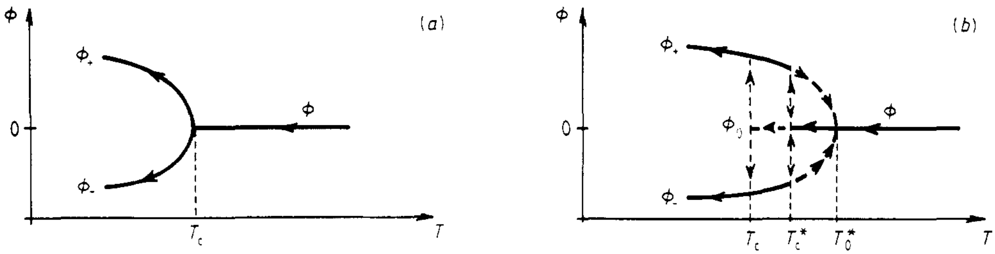

Figure 3. The two types of bifurcations for ϕ as a function of temperature when $\phi = \phi^*$ ((a) $A_4 > 0$, (b) $A_4 < 0$). The full curves indicate reversible processes for stable phases and the broken lines indicate irreversible processes for metastable phases.

branches of ϕ for $\phi \neq 0$ and $\phi_0 = 0$, full curves indicate absolutely stable solutions, broken lines indicate metastable solutions and arrows indicate the direction of path along the temperature axis. Since $A_4 < 0$ corresponds to a first-order phase transition, the temperature hysteresis of the solution shown in figure 3(b) is understandable. It should be noted that at the tricritical point $T_0^* \to T_c^* \to T_c$, and hence figure 3(b) becomes identical with figure 3(a). In the general case when ϕ is independent of ϕ^* the sign reversal can be considered a phase rotation by an angle $\pm\pi$. In fact, any phase rotation $\psi \to \psi + \delta\psi$ where

$$\phi = \eta \exp(\mathrm{i}\psi) \tag{3}$$

is an invariance transformation for the Lagrangian $\mathscr{L}$ of equation (1), as shown schematically in figure 4. This also means that the vacuum state of the system is

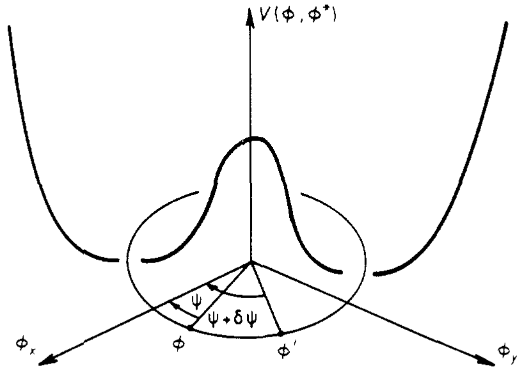

Figure 4. Schematic illustration of the rotational invariance of the potential $V(\phi, \phi^*)$. We use the notation: $\phi = \phi_x + \mathrm{i}\phi_y$.

infinitely degenerate. In all cases, however, the broken symmetry of the Lagrangian results in bifurcations of the stable solution.

2. The equation of motion

Using the Euler-Lagrange equation

$$\nabla \frac{\partial \mathscr{L}}{\partial(\nabla\phi^*)} + \frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \phi_t^*} - \frac{\partial \mathscr{L}}{\partial \phi^*} = 0 \tag{4}$$

we readily obtain the fundamental equation of motion for the complex order-parameter field ϕ in the form of a non-linear Klein-Gordon equation:

$$\Box_\varepsilon \phi = -2[A_2 + 2A_4\phi\phi^* + 3A_6(\phi\phi^*)^2]\phi \tag{5}$$

where

$$\Box_\varepsilon \equiv \frac{\partial^2}{\partial x_0^2} + \varepsilon \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \qquad \varepsilon = \pm 1$$

is the Laplace-Beltrami operator, $\varepsilon = -\mathrm{sgn}(D)$, $x_0 = m^{-1/2}t$ and $(x_1, x_2, x_3) = |D|^{-1/2}(x, y, z)$. We then use the polar representation of equation (3) for ϕ where η is called the envelope and ψ the carrier wave and both of them are real functions of (x_0, x_1, x_2, x_3). Substituting the polar form of ϕ given by (3) into (5) and separating the real and imaginary parts yields the following equivalent system of partial differential equations (PDE) for η and ψ:

$$\Box_\varepsilon \eta - (\nabla\psi|\nabla\psi)\eta = -2(A_2 + 2A_4\eta^2 + 3A_6\eta^4)\eta \tag{6}$$

$$2(\nabla\eta|\nabla\psi) + \eta\Box_\varepsilon\psi = 0 \tag{7}$$

where $\nabla \equiv (\partial/\partial x_0, \partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ is the $(3+1)$-dimensional gradient operator and $(x|y) \equiv x_0y_0 + \varepsilon\, \Sigma_{i=1}^3\, x_iy_i$ is a scalar product defined with respect to the signature.

We intend to solve the system of equations (6) and (7) by effectively decoupling them through the imposition of the ansatz

$$(\nabla\psi|\nabla\psi) = \alpha + \beta\eta^2 + \gamma\eta^4 \tag{8}$$

where α, β and γ are adjustable parameters. This yields the equation for the envelope in the form of a non-linear Klein-Gordon equation:

$$\Box_\varepsilon \eta = -2(a_2\eta + 2a_4\eta^3 + 3a_6\eta^5) \tag{9}$$

where $a_2 = \frac{1}{2}\alpha + A_2$, $a_4 = \frac{1}{4}\beta + A_4$ and $a_6 = \frac{1}{6}\gamma + A_6$. Then the carrier wave must satisfy an overdetermined system of equations:

$$\Box_\varepsilon \psi = 0 \qquad (\nabla\eta \,|\, \nabla\psi) = 0 \tag{10}$$

together with equation (8). A systematic analysis of solutions of these equations from the viewpoint of the symmetry groups can be obtained using the symmetry reduction method (Ibragimov 1985, Olver 1986, Ovsiannikov 1982, Sattinger and Weaver 1986, Grundland *et al* 1984).

Here, we only concern ourselves with the case where the PDE (9) can be reduced to an ODE. The method used consists essentially of four steps (for details see Winternitz *et al* 1987).

(i) Find the symmetry group G of equation (9) and its Lie algebra L. It has been proved that in Minkowski and Euclidean spaces equation (9) is invariant with respect to the Poincaré group P(3, 1) and the Euclidean group E(4), respectively. In the special case when the coefficients $a_2 = a_4 = 0$, the symmetry group of equation (9) is even larger. In four-dimensional space the equation is invariant with respect to the similitude group, i.e. Poincaré P(3, 1) or Euclidean E(4) extended by dilations which are denoted by Sim(3, 1) and Sim(4), respectively. In three-dimensional space when $a_2 = a_4 = 0$, the symmetry group of equation (9) also contains conformal transformations and is denoted by Conf(2, 1) and Conf(3). This group is locally isomorphic to the de Sitter groups O(3, 2) and O(4, 1), respectively.

(ii) Find all subalgebras $L_i \subset L$ and all subgroups $G_i \subset G$ having generic orbits of codimension one in the space of independent variables (x^μ). The relevant classification of subgroups was made by Patera *et al* (1977).

(iii) Find the invariants of each subgroup G_i in the space of independent and dependent variables (x^μ, ϕ), i.e. find the first integrals of the system of first-order PDE:

$$X_a H(x, \phi) = 0 \qquad 1 \leqslant a \leqslant k \tag{11}$$

where $\{X_1, \ldots, X_k\}$ is a basis of the given subalgebras L_i and X_a are linear first-order operators. Various subgroups allow us to construct different solutions of the form

$$\phi(\boldsymbol{x}) = \rho(\boldsymbol{x}) F(\xi(\boldsymbol{x})) \tag{12}$$

where ρ and ξ are functions of $\boldsymbol{x}$ given by the symmetry of the problem, and F is a function of ξ only, which is subject to the reduced ODE. Passing systematically through all subalgebras G_i we obtain all so-called symmetry variables ξ and the corresponding ODE for F.

(iv) Find the solutions of each ODE. These ODE can often be explicitly integrated in terms of known functions, or at least, their singularity structure can be investigated using well known methods.

3. Solutions of the equation of motion

3.1. The carrier wave

In this section we intend to obtain solutions to the system of equations for the carrier wave, equations (8) and (10).

First of all, it is easy to see that a simple way of satisfying these conditions on ψ is to set $\beta=\gamma=0$ in (8) and to demand that η and ψ depend on different sets of independent variables. This means that we have the following three cases: (i) $\psi=\psi(x_i)$ and $\eta=\eta(x_j, x_k, x_l)$; (ii) $\psi=\psi(x_i, x_j)$ and $\eta=\eta(x_k, x_l)$; (iii) $\psi=\psi(x_i, x_j, x_k)$ and $\eta=\eta(x_l)$. In each case (x_i, x_j, x_k, x_l) are supposed to exhaust all inequivalent permutations of the independent variables (x_0, x_1, x_2, x_3). This results in two general types of equations for the carrier wave: either the Laplace equation or the wave equation. Depending on whether we deal with case (i), (ii) or (iii) these are one-, two- or three-dimensional PDE. We now discuss explicit solutions of these equations subject to the constraint of (8) with $\beta=\gamma=0$.

(i) In one-dimensional cases we have

$$\psi(x_i)=\sqrt{\varepsilon\alpha}\, x_i+\alpha_0 \tag{13}$$

where $\varepsilon=\pm1$ and α_0 is an integration constant.

(ii) In two-dimensional cases the solutions of the resultant Laplace equation are

$$\psi(x_i, x_j)=f_1(x_i+\mathrm{i}x_j)+f_2(x_i-\mathrm{i}x_j) \tag{14}$$

where f_1 and f_2 are arbitrary twice differentiable functions which are related via

$$f_2(x)=\frac{\alpha}{4}\int\left(\frac{\mathrm{d}f_1}{\mathrm{d}x}\right)^{-1}\mathrm{d}x. \tag{15}$$

If the resultant equation is a two-dimensional wave equation, then the solution for ψ is the well known d'Alembert solution

$$\psi(x_i, x_j)=f_1(x_i+x_j)+f_2(x_i-x_j) \tag{16}$$

where f_1 and f_2 are of the same class as before and they are also related through equation (15).

(iii) In three-dimensional cases the solutions of the resultant Laplace equation with this constraint can only be of translation-wave form (see Cieciura and Grundland 1984), i.e.

$$\psi(x_i, x_j, x_k)=\sum_{\alpha=i,j,k}(\mu_\alpha x_\alpha+\nu_\alpha) \tag{17}$$

where μ_α, ν_α are constants. The $(2+1)$-dimensional wave equation subject to this constraint admits a much more interesting class of solutions, namely (see Grundland *et al* 1984)

$$\psi(x_0, x_j, x_k)=x_k+g(x_0+x_j) \tag{18}$$

where g is an arbitrary function. This solution is related to the so-called degenerate symmetry variables existing in M(2, 1).

We may now return to the original equations (8) and (10) and relax the condition that $\beta=\gamma=0$. The following analysis is based on the symmetry reduction method and, in particular, on the results published by Grundland *et al* (1984) for subgroups of codimension 2. Suppose that

$$\eta=\eta(\xi_1(x)) \qquad \psi=\psi(\xi_2(x)). \tag{19}$$

We then wish to find the conditions and the form of ξ_1 and ξ_2 so that η and ψ satisfy equations (8)-(10). First of all

$$\Box_\varepsilon\psi=(\nabla\xi_2|\nabla\xi_2)\psi''+(\Box_\varepsilon\xi_2)\psi'=0 \tag{20}$$

which can be satisfied if $\Box_\varepsilon\xi_2=0$ and $(\nabla\xi_2|\nabla\xi_2)=\Sigma_{n=0}\,\lambda_n\xi_1^n$ where λ_n are constants.

Provided at least one of the coefficients λ_n is non-zero, we immediately obtain the form of ψ as

$$\psi = \mu\xi_2 + \nu \tag{21}$$

where μ and ν are constants. Another possibility arises when $\Box\xi_2 = f(\xi_2)$ and $\lambda_0 \neq 0$ together with $\lambda_n = 0$, $n \geq 1$. This leads to

$$\psi(\xi_2) = \int \mathrm{d}\xi_2 \exp\left(-\frac{c_1}{\lambda_0}\int f(\xi_2)\,\mathrm{d}\xi_2\right) + c_2 \tag{22}$$

where c_1 and c_2 are integration constants. In particular, if $f(\xi_2) = \delta_2/\xi_2$, then

$$\psi = c_1\xi_2^{1-\delta_2/\lambda_0} + c_2 \qquad \text{if } \delta_2/\lambda_0 \neq 1 \tag{23}$$

or

$$\psi = c_1 \ln \xi_2 + c_2 \qquad \text{if } \delta_2/\lambda_0 = 1. \tag{24}$$

Secondly,

$$(\nabla\eta|\nabla\psi) = \eta'\psi'(\nabla\xi_1|\nabla\xi_2) = 0 \tag{25}$$

which can obviously be satisfied if $(\nabla\xi_1|\nabla\xi_2) = 0$. Thirdly,

$$\begin{aligned}(\nabla\psi|\nabla\psi) = \psi'^2(\nabla\xi_2|\nabla\xi_2) &= \mu^2 \sum_{n=0} \lambda_n\xi_1^n \\ &= \alpha + \beta\eta^2(\xi_1) + \gamma\eta^4(\xi_1).\end{aligned} \tag{26}$$

If $\beta = \gamma = 0$, then the above condition does not constrain the form of the functional dependence of η on ξ_1 and it leads to $\lambda_n = 0$, $n \geq 1$ and $\mu^2\lambda_0 = \alpha$. If, on the other hand, $\beta \neq 0$ or $\gamma \neq 0$, then this necessarily is followed by $\eta = |\mu|(\lambda_2/\beta)^{1/2}$, $\gamma = \lambda_4\beta^2/\lambda_2^2\mu^2$ and $\lambda_1 = \lambda_3 = 0$. Finally,

$$\begin{aligned}\Box_\varepsilon\eta &= (\nabla\xi_1|\nabla\xi_1)\eta'' + (\Box_\varepsilon\xi_1)\eta' \\ &= -2(a_2\eta + 2a_4\eta^3 + 3a_6\eta^5)\end{aligned} \tag{27}$$

can be satisfied if $\Box_\varepsilon\xi_1 = 0$ and $(\nabla\xi_1|\nabla\xi_1) = \omega$ where ω is a constant. In this case equation (27) can be directly integrated to yield

$$\eta'^2 = (-1/\omega)(a_2\eta^2 + a_4\eta^4 + a_6\eta^6) \tag{28}$$

provided $\omega \neq 0$. This leads to a very large class of solutions in terms of elementary and elliptic functions. For an exhaustive analysis the reader is referred to the paper of Winternitz *et al* (1987). If, however, $\omega = 0$, then $\Box_\varepsilon\xi_1$ is not necessarily zero, but an arbitrary function of $\xi_1 : \Box_\varepsilon\xi_1 = g(\xi_1)$. Then

$$\int \mathrm{d}\eta[-2\eta(a_2 + 2a_4\eta^2 + 3a_6\eta^4)]^{-1/2} = \int \mathrm{d}\xi_1\, g(\xi_1) \tag{29}$$

yielding an explicit form of $\eta = \eta(\xi_1)$. If, in particular, $g(\xi_1) = \delta_1/(\xi_1 + \xi_0)$, then

$$-4a_2\delta_1 \ln(\xi_1 + \xi_0) = \ln\frac{\eta^2}{(3a_6\eta^4 + 2a_4\eta^2 + a_2)^{1/2}} - a_4h(\eta) \tag{30}$$

where $\Delta \equiv a_4^2 - 3a_2a_6$ and

$$h(\eta) = \begin{cases} \dfrac{1}{\sqrt{\Delta}} \tan^{-1} \dfrac{3a_6\eta^2 + a_4}{\sqrt{\Delta}} & \text{if } \Delta > 0 \\ \dfrac{1}{2\sqrt{-\Delta}} \ln \dfrac{-3a_6\eta^2 - a_4 + \sqrt{-\Delta}}{3a_6\eta^2 + a_4 + \sqrt{-\Delta}} & \text{if } \Delta < 0 \\ -(a_4 + 3a_6\eta^2)^{-1} & \text{if } \Delta = 0. \end{cases} \tag{31}$$

To summarise, we have been looking for $\eta = \eta(\xi_1)$, $\psi = \psi(\xi_2)$ and ξ_1, ξ_2 such that the following equations for the symmetry variables are satisfied:

$$\Box \xi_1 = \delta_1/\xi_1 \qquad \Box \xi_2 = \delta_2/\xi_2 \tag{32}$$

$$(\nabla \xi_1 | \nabla \xi_2) = 0 \tag{33a}$$

$$(\nabla \xi_1 | \nabla \xi_1) = \omega \tag{33b}$$

$$(\nabla \xi_2 | \nabla \xi_2) = \sum_{n=0} \lambda_n \xi_1^n. \tag{33c}$$

If $\delta_2 = 0$, then $\psi(\xi_2)$ is given by equation (21), otherwise ψ is given by (23) and (24). If $\delta_1 = 0$, then $\eta(\xi_1)$ is given by equation (28), otherwise η is given implicitly by (30). We have analysed table IV of Grundland *et al* (1984) for codimension-2 symmetry variables in M(3, 1) and selected the appropriate entries that satisfy our equation (33). It was easy to supplement it with the relevant solutions in E(4). These results are summarised in table 1. Obviously, the spatial variables x_1, x_2 and x_3 are entirely equivalent, hence they can be permuted amongst themselves in the definitions of ξ_1 and ξ_2 leading to a larger number of possibilities than just those shown in table 1. Most of the entries in table 1 (all except for numbers II 10, 11, 12, 27) list ξ_1 and ξ_2 which depend on different sets of variables and have already been recognised. In most cases $\omega \neq 0$ and hence the envelope η is to be found from equation (28). The other possibility, $\omega = 0$, is present only in numbers II 4, 17, 27 and it leads to η given by equation (30). Finally, only numbers II 12 and II 27 have a non-zero λ_i where $i > 0$.

Having discussed the carrier wave equation and its solutions we now turn to the envelope equation, equation (9).

3.2. *The envelope*

The non-linear Klein-Gordon equation, equation (9), written for a scalar field η has recently been extensively studied by Winternitz *et al* (1987). A large number of new exact solutions have been found using the method of symmetry reduction in (3+1)-dimensional Euclidean ($\varepsilon = +1$) or Minkowski ($\varepsilon = -1$) space of independent variables. In order to adapt the obtained results to the present situation we must include the solutions η which either depend on an incomplete set of variables: $\eta(x_i)$, $\eta(x_i, x_j)$, $\eta(x_i, x_j, x_k)$, or depend on the symmetry variables ξ_1, ξ_2 of table 1.

In the Euclidean case when a_2 and a_4 are not simultaneously equal to zero the symmetry variables of the envelope can only be given by

$$\xi = (x_0^2 + x_1^2 + \ldots + x_k^2)^{1/2} \qquad k = 0, 1, 2$$

which corresponds to space-independent and time-dependent solutions ($k = 0$), rectilinear propagation ($k = 1$) and cylindrical propagation ($k = 2$).

Table 1. The results of symmetry reduction for ξ_1 and ξ_2 of equation (39) following Grundland *et al* (1984).

Number		ξ_1		ξ_2	$\delta_1=\xi_1\Box\xi_1$	$\delta_2=\xi_2\Box\xi_2$	$\omega=(\nabla\xi_1\|\nabla\xi_1)$	$(\nabla\xi_2\|\nabla\xi_2)$	δ_2/λ_0
I	1	x_0	E(4)	$(x_1^2+x_2^2+x_3^2)^{1/2}$	0	2	1	1	2
	2	$(x_0^2+x_1^2)^{1/2}$		$(x_2^2+x_3^2)^{1/2}$	1	1	1	1	1
	3	x_0		$(x_2^2+x_3^2)^{1/2}$	0	1	1	1	1
	4	x_0		x_1	0	0	1	1	0
II	1	x_1	M(3, 1)	x_2	0	0	-1	-1	0
	2	x_0		x_1	0	0	1	-1	0
	3	x_1		x_0	0	0	-1	1	0
	4	x_0+x_1		x_2	0	0	0	-1	0
	5	x_2		x_0+x_1	0	0	-1	0	—
	6	x_2		$(x_0^2-x_1^2)^{1/2}$	0	1	-1	1	1
	7	$(x_0^2-x_1^2)^{1/2}$		x_2	1	0	1	-1	0
	8	x_0		$(x_1^2+x_2^2)^{1/2}$	0	-1	1	-1	1
	9	$(x_1^2+x_2^2)^{1/2}$		x_0	-1	0	-1	1	0
	10	$x_0+a\sin^{-1}[x_1/(x_1^2+x_2^2)^{1/2}]$		$(x_1^2+x_2^2)^{1/2}$	0	-1	1	-1	1
	11	$(x_1^2+x_2^2)^{1/2}$		$x_0+a\sin^{-1}[x_1/(x_1^2+x_2^2)^{1/2}]$	-1	0	-1	1	0
	12	$x_2+\frac{1}{4}(x_0+x_1)^2$		$x_0-x_1+(x_0+x_1)x_2+\frac{1}{6}(x_0+x_1)^3$	0	0	-1	$4\xi_1$	—
	13	x_3		$(x_1^2+x_2^2)^{1/2}$	0	-1	-1	-1	1
	14	$(x_1^2+x_2^2)^{1/2}$		x_3	-1	0	-1	-1	0
	15	$(x_0^2-x_1^2)^{1/2}$		$(x_2^2+x_3^2)^{1/2}$	1	-1	1	-1	1
	16	$(x_2^2+x_3^2)^{1/2}$		$(x_0^2-x_1^2)$	-1	1	-1	1	1
	17	x_0+x_1		$(x_2^2+x_3^2)^{1/2}$	0	-1	0	-1	1
	18	$(x_2^2+x_3^2)^{1/2}$		x_0+x_1	-1	0	-1	0	—
	19	x_3		$(x_0^2-x_1^2-x_2^2)^{1/2}$	0	2	-1	1	2
	20	$(x_0^2-x_1^2-x_2^2)^{1/2}$		x_3	2	0	1	-1	0
	21	x_3		$x_2+\frac{1}{4}(x_0+x_1)^2$	0	0	-1	-1	0
	22	$x_2+\frac{1}{4}(x_0+x_1)^2$		x_3	0	0	-1	-1	0
	23	x_3		$x_2+a\ln(x_0+x_1)$	0	0	-1	-1	0
	24	$x_2+a\ln(x_0+x_1)$		x_3	0	0	-1	-1	0
	25	x_0		$(x_1^2+x_2^2+x_3^2)^{1/2}$	0	-2	1	-1	2
	26	$(x_1^2+x_2^2+x_3^2)^{1/2}$		x_0	-2	0	-1	1	0
	27	x_0+x_1		$x_2+\varepsilon(x_0+x_1)x_3$	0	0	0	$-1-\xi_1^2$	0

In the Minkowski case when a_2 and a_4 are not simultaneously equal to zero the symmetry variables of the envelope involve a larger number of choices, namely

$$\xi = (x_0^2 - x_1^2 - \ldots - x_k^2)^{1/2} \qquad k = 0, 1, 2$$

$$\xi = (x_1^2 + x_2^2 + \ldots + x_{k+1}^2)^{1/2} \qquad k = 0, 1, 2$$

$$\xi = x_1 + p \ln(x_0 + x_2)$$

$$\xi = x_1 + \tfrac{1}{4}(x_0 + x_2)^2$$

and a degenerate variable of the form

$$\xi = x_1 + \phi(x_0 + x_2)$$

where ϕ is an arbitrary function, which is a generalisation of the last two cases. The profiles of the corresponding solutions in both Minkowski and Euclidean cases include constant solutions (mean fields), singular solutions with one and with two singularities (defect structures), kinks (domain walls), bumps (nucleation centres) and Jacobi elliptic functions (elementary excitations). For analytical forms of these functions the reader is referred to the paper of Winternitz *et al* (1987).

When $a_2 = a_4 = 0$, the solutions obtained differ significantly from the previous cases. A large number of algebraic, trigonometric, hyperbolic and Jacobi elliptic solutions have been found which possess interesting propagation properties and singularity structures. The relevant symmetry variables of the envelope lead to unusual surfaces of constant phase such as ellipsoids, hyperboloids and helicoidal surfaces which may be useful in physical applications. The singularity structures exhibit various nucleation effects involving, for example, point and line defects growing to become spherical and cylindrical defects, respectively.

4. A bifurcating solution

In this section we intend to present and analyse a particular exact solution of the envelope equation, equation (9), which is physically interesting as an example of bifurcating solution. Its explicit form is

$$\eta(x) = [-8a_6(x_1^2 + x_2^2)]^{-1/4} \left(\frac{z_k + 1}{z_k - 2} \right)^{1/2} \tag{34}$$

where

$$\begin{aligned} z_k &= \cosh[\tan^{-1}(x_2/x_1 - \lambda_0) + 2k\pi] \\ &= \cos\left[\tfrac{1}{2} \ln\left(\frac{1 - \mathrm{i}(x_2/x_1 - \lambda_0)}{1 + \mathrm{i}(x_2/x_1 - \lambda_0)} \right) + 2\mathrm{i}k\pi \right] \end{aligned} \tag{35}$$

with $k = 0, \pm 1, \pm 2, \ldots$ and λ_0 being a complex integration constant. From § 3.1 the corresponding carrier wave is

$$\psi(x_0, x_3) = f_1(x_0 + \sqrt{\varepsilon}\, x_3) + f_2(x_0 - \sqrt{\varepsilon}\, x_3)$$

with f_1 and f_2 related through equation (15) and $\varepsilon = -\mathrm{sgn}(D)$. The envelope function of equation (35) exhibits interesting bifurcation properties which we now discuss in detail. Notice first that the function ln( ) is a univalent analytical function in a simply connected region which does not contain the origin and infinity. At the origin and at infinity it has branching points of infinite order. Every branch of the logarithm differs by $2k\pi\mathrm{i}$, so we denote each branch as η_k. To determine this function in the entire complex plane

$$z \equiv \frac{1-\mathrm{i}(x_2/x_1-\lambda_0)}{1+\mathrm{i}(x_2/x_1-\lambda_0)}$$

a cut along any line joining the origin and infinity is introduced. Thus the function ln( ) has bifurcation points of infinite order whenever

$$x_2/x_1-\lambda_0 \pm \mathrm{i}=0.$$

This means that the solution η_k bifurcates for

$$x_2=x_1 \operatorname{Re} \lambda_0 \qquad \operatorname{Im} \lambda_0= \pm 1. \tag{36}$$

Similarly, the function $\sqrt{(\;)}$ is a univalent analytical function in a simply connected region which does not contain the origin and infinity. At the origin and at infinity it has bifurcation points of second order. The two branches of the square root differ only by the sign. To examine this function in the entire complex plane

$$w \equiv (z_k+1)/(z_k-2)$$

a cut along any line joining the origin and infinity is introduced. Thus the function $\sqrt{w}$ has bifurcation points of second order whenever $z_k+1=0$. This means that the solution η_k bifurcates for

$$x_2=x_1 \operatorname{Re} \lambda_0 \qquad \operatorname{Im} \lambda_0=(1-E_l)/(1+E_l) \tag{37}$$

where $E_l \equiv \exp[2\pi(1+2l)]$. Moreover, the solution given by equation (34) has simple poles at either $x_1=x_2=0$ or $z_k-2=0$. Therefore η_k diverges for

$$x_1=x_2=0 \qquad \text{or} \qquad x_2=x_1(3.85+\operatorname{Re} \lambda_0) \qquad \operatorname{Im} \lambda_0=0. \tag{38}$$

In order to facilitate physical interpretation of the properties of this solution, it is convenient to represent equation (34) in polar form as

$$\eta_{n,k}=\rho_{n,k} \exp(\mathrm{i}\psi_{n,k}) \tag{39}$$

where the amplitude is given by

$$\rho_{n,k}=1+\frac{3(2 \cos I \cosh R-1)}{\cos^2 I-\cosh^2 R-4 \cos I \cosh R} \tag{40}$$

and the phase is given by

$$\psi_{n,k}=\tfrac{1}{2}\pi(n+\tfrac{1}{2})+\tfrac{1}{2}\tan^{-1}\left(\frac{-3 \sin I \sinh R}{-\sin^2 I \cosh 2R+\sinh^2 R-\cos I \cosh R}\right) \tag{41}$$

where $n=0$, 1, 2, 3 and we have denoted

$$R \equiv 2k\pi+\frac{x_2/x_1-\operatorname{Re} \lambda_0}{(x_2/x_1-\operatorname{Re} \lambda_0)^2+(\operatorname{Im} \lambda_0)^2-1} \tag{42}$$

and

$$I=-\tfrac{1}{2}\ln\left(\frac{-4 \operatorname{Im} \lambda_0}{(x_2/x_1-\operatorname{Re} \lambda_0)^2+(\operatorname{Im} \lambda_0)^2+2 \operatorname{Im} \lambda_0+1}\right). \tag{43}$$

Note that the new index n is used to label the four values of the fourth root of -1, i.e. $\exp[\mathrm{i}\frac{1}{2}\pi(n+\frac{1}{2})]$.

In a similar way, we can introduce polar notation in the (x_2, x_1) plane. Denoting

$$r^2 = x_1^2 + x_2^2 \qquad \tan\alpha = x_2/x_1 - \lambda_0 \tag{44}$$

we can express $\eta_{n,k}$ equivalently as

$$\eta_{n,k} = \exp[\mathrm{i}\tfrac{1}{2}\pi(n+\tfrac{1}{2})](8a_6r^2)^{-1/4}\left(\frac{\cosh(\alpha+2k\pi)+1}{\cosh(\alpha+2k\pi)-2}\right)^{1/2}. \tag{45}$$

From equation (37) the angle at which η bifurcates is

$$\alpha_0^l \equiv \tan^{-1}(\mathrm{Re}\,\lambda_0) + (2l+1)\pi \tag{46}$$

while from equation (38) the angle at which η diverges is

$$\alpha_\infty^l = \tan^{-1}(3.85 + \mathrm{Re}\,\lambda_0) + (2l+1)\pi. \tag{47}$$

It is now obvious that both the amplitude of the solution ρ and its phase ψ depend on the polar angle α in the space of independent variables (x_1, x_2). In the regions where η is analytical, i.e. for $\alpha \neq a_0^l$, $\alpha \neq \alpha_\infty^l$, we can uniquely represent the amplitude and the phase as functions of α:

$$\rho_{n,k} = \rho_{n,k}(\alpha) \qquad \psi_{n,k} = \psi_{n,k}(\alpha). \tag{48}$$

Moreover, the converse is also true: we can write the polar angle α in a unique way as a function of the phase ψ, $\alpha = \alpha(\psi_{n,k})$, for each interval where $\eta_{n,k}$ is analytical. One can obtain an explicit formula for this function by solving a quartic equation in terms of $\exp(R)$ and express η as

$$\eta_{n,k} = \rho_{n,k}[\alpha(\psi_{n,k})]\exp[\mathrm{i}\alpha(\psi_{n,k})]. \tag{49}$$

This represents the solution, equation (34), in a polar form analogous to that of equation (48), i.e. where the amplitude and the phase are expressed in terms of dependent variables. A rotation of ψ by $\delta\psi$, as argued before, is an invariance transformation for the Lagrangian. As shown in equation (49), this is definitely not an invariance transformation for the solution. In the regions where $\eta_{n,k}$ is analytic both the phase and the amplitude change continuously with the rotation of the polar angle α. At the bifurcation angles α_0^l, however, both the phase $\psi_{n,k}$ and the amplitude $\rho_{n,k}$ are multi-valued functions. Since both $\psi_{n,k}$ and $\rho_{n,k}$ depend on α through hyperbolic functions (see equations (40)-(43)), a shift to another branch caused by a $2\pi l$ rotation in the (x_1, x_2) plane produces an essentially different function $\psi_{n,k+l}$.

The bifurcation phenomenon which we have described for the particular solution $\eta_{n,k}$ of equation (34) can now be compared with the general picture shown in figure 3 for an arbitrary solution. This has been illustrated in figure 5. Whereas the homogeneous field ϕ bifurcates at the critical temperature T_c, resulting in a phase-rotation degeneracy (with a non-denumerable number of equivalent states), our solution of equation (34) which exists at a fixed temperature T_c^* bifurcates as a result of the rotation in the (x_1, x_2) plane (with a denumerable number of equivalent states). The broken symmetry of the Lagrangian is not only the phase-rotation symmetry but also the spatial rotation symmetry. The latter is much less obvious than the former.

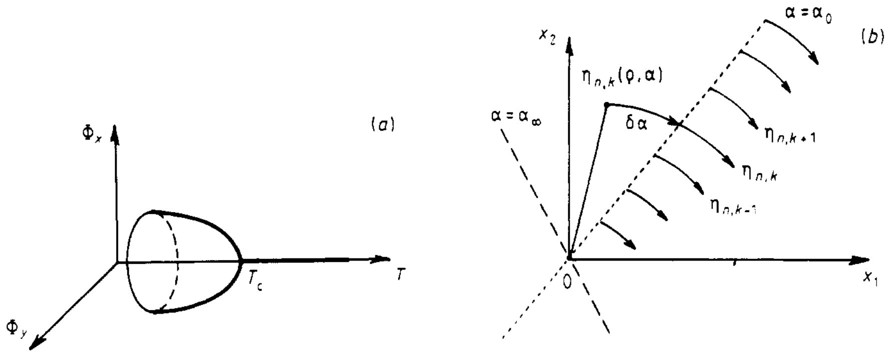

Figure 5. Schematic illustration of the analogy between bifurcations of the general solution ϕ and the particular solution $\eta_{n,k}$ of equation (34). Broken symmetry: (a) phase rotation, (b) spatial rotation.

The mapping between the real (physical) space and the order-parameter space is of crucial importance to the theory of defect structures (singularities in the phase space) (Anderson 1984). A defect or a singularity is allowed to exist since the order parameter may assume its equilibrium value everywhere except possibly in a region which is of a topologically lower dimension than that of real space. For example, point, line and plane defects are allowed in three-dimensional space. Thus equation (34), which represents a plane singularity, can be considered a legitimate defect structure in three-dimensional space.

It should also be noted that as a result of the relationship between the coefficients a_2, a_4 and a_6 of equation (9) and the coefficients A_2, A_4 and A_6 of equation (1), the temperature range of existence of solitary-wave envelopes (see § 3.2) may be significantly altered. Consequently, some solutions which for real ϕ are confined to a certain temperature, e.g. the tricritical temperature, may for complex ϕ exist also at other temperatures in its neighbourhood provided $\alpha \neq 0$ or $\beta \neq 0$. This would be an indication of the influence of the carrier wave on the envelope.

Finally, it is of interest to examine the structural stability conditions for the solutions obtained using the Lagrangian of equation (1) or, alternatively, the Hamiltonian of equation (2). The solutions listed here have been derived so as to extremise the Hamiltonian functional, i.e. $\delta H = 0$. They will correspond to local minima of this functional if and only if the second variation of H is positive, i.e. for $\delta^2 H > 0$. Assuming, as we have done before, that the form of ϕ is $\phi = \eta(\xi_1)\exp[i\psi(\xi_2)]$ and utilising the definitions of x_0 and x_1, x_2, x_3, we readily find the resultant conditions for the quadratic form in the second variation to be positive definite as

$$(\partial\xi_1/\partial x_0)^2 + \mathrm{sgn}(D)|\nabla\xi_1|^2 > 0 \tag{50a}$$

$$(\partial\xi_2/\partial x_0)^2 + \mathrm{sgn}(D)|\nabla\xi_2|^2 > 0. \tag{50b}$$

These two inequalities must be satisfied simultaneously by ξ_1 and ξ_2 in order for ϕ to correspond to a local minimum of H. It is easy to see that all possible combinations of ξ_1 and ξ_2 satisfy equation (50) in the Minkowski case M(3, 1) since there $D > 0$. Using table 1 we also find that no solutions in the Euclidean case E(4) are capable of satisfying equation (50). We therefore conclude that for $D < 0$ the solutions obtained in this paper are unstable.

5. Possible applications

In this section we briefly describe possible applications of the presented model to several critical phenomena. The model calls for the use of a complex order parameter and hence qualifies as a description of, for example, the superfluid–viscous fluid phase transition, the superconducting–normal metal phase transition, the cholesteric–nematic liquid crystal phase transition, or the transitions involving helicoidal magnetic structures. Rather than use the microscopic Lagrangian density of equation (1) or its corresponding Hamiltonian density (the Landau–Ginzburg–Wilson Hamiltonian), it has been customary to make physical applications of the ϕ^4 or ϕ^6 models based on a related phenomenological free energy density. Its form is usually adopted as (White and Geballe 1979, Luban 1976)

$$F = A_2|\phi|^2 + A_4|\phi|^4 + A_6|\phi|^6 + \tfrac{1}{2}D|\nabla\phi|^2. \tag{51}$$

This type of series expansion neglects kinetic effects by dropping the term $\frac{1}{2}m\phi_t^2$. This aspect has recently been pointed out by Pippard (1987) who found important repercussions for entropy calculations, especially in finite-size systems. The equation of motion (the Euler equation) for ϕ can be derived by setting the functional derivative of F to zero, $\delta F/\delta\phi^* = 0$, which yields

$$A_2\phi + 2A_4|\phi|^2\phi + 3A_6|\phi|^4\phi - \tfrac{1}{2}D\Delta\phi = 0 \tag{52}$$

and has the form of our equation (4) in three-dimensional Euclidean space. If one followed the advice of Pippard (1987) and included the kinetic energy, the agreement between the two approaches would be perfect. The approach based on the Hamiltonian or Lagrangian density so far has been mostly adopted in field theory (Jackiw 1977) while free-energy expansions have been widely used in various areas of condensed matter physics.

In the remainder of this section we shall briefly discuss several prominent physical systems to which the free-energy expansion of equation (51) has been applied in the past. The inclusion of the sixth-order term allows us to embrace both second- and first-order transitions using the same potential energy form.

5.1. *Superfluidity*

The λ transition of liquid ^{4}He is described (Luban 1976) using the so-called Ginzburg–Pitaevskii thermodynamic potential which is of the form of equation (51) where: $A_2 = a(T - T_c)$, $A_4 = \frac{1}{2}b$ and $D = \hbar^2/2m$ with m denoting the mass of the ^{4}He atom. The order parameter $\phi = \eta\exp(i\psi)$ is the effective wavefunction of the superfluid component, and the superfluid mass density is $\rho_s = m\eta^2$. In order to obtain the correct scaling of ρ_s, Mamaladze (1967) postulated that

$$A_2 = a\,\mathrm{sgn}(\varepsilon)|\varepsilon|^{4/3} + \tfrac{1}{2}mv_s^2 \qquad A_4 = \tfrac{1}{2}b|\varepsilon|^{2/3}$$

where v_s is the critical velocity and $\varepsilon = (T - T_c)/T_c$.

5.2. *Superconductivity*

In this case the Ginzburg–Landau model adopts equation (51) with $D = \hbar^2/2m^*$ and the order parameter is chosen to be the superconducting wavefunction $\Phi(\boldsymbol{r})$ (or the pair potential $\Delta(\boldsymbol{r})$). Its square is the density of superconducting electrons $n_s(\boldsymbol{r}) =$

$|\Phi(\boldsymbol{r})|^2$. The superconducting coherence length ξ is proportional to $(A_2)^{-1/2}$. It can also be found (White and Geballe 1979) that the superconducting current density j_s in the absence of external magnetic fields is proportional to the density of superconducting electrons and the gradient of the phase, i.e.

$$j_s = \frac{e^*\hbar}{m^*}|\Phi|^2\nabla\psi$$

where e^* is the effective electron charge, m^* is the effective electron mass and $\hbar$ is Planck's constant. Hence, based on § 3.1 we would find various superconducting currents for different choices of the carrier wave ψ. The bifurcating solution of equation (34) can be interpreted as a planar distribution of superconducting electrons with a field surrounding it.

5.3. *Liquid crystals*

Here, the order parameter can be chosen as an ordering tensor $S_{ij}(\boldsymbol{r})$ (see, e.g., Warner 1984) which is characterised both by its magnitude and direction. The free energy analogous to equation (51) has been used for liquid crystals by Haken (1974). The fundamental equation of motion of the type given by equation (52) has been used in the past in this context (Warner 1984, Guyon 1975, Lei *et al* 1985). Our bifurcating solution, equation (34), fits particularly well as a layered, spontaneously twisted helicoidal structure arising during a nematic-cholesteric phase transition. The nematic liquid crystal is an anisotropic fluid made of rod-like molecules aligned along one direction without the centres of gravity of the molecules being ordered. A cholesteric phase has no mirror symmetry but has local nematic order. On a large scale, the order parameter forms layered structures (de Gennes 1974).

5.4. *Helicoidal metamagnets*

Helicoidal metamagnets (Herpin and Mériel 1961) can be considered defect structures with respect to a homogeneous distribution of the magnetisation field which plays the role of an order parameter. They consist of a layered distribution of parallel planes of magnetisation. The orientation of the magnetisation vector in each plane is shifted with respect to its neighbours by a constant angle. In the continuum limit, the free-energy density, equation (51), is a suitable choice for these structures provided we interpret ϕ as

$$\phi = \eta\, e^{i\psi} \qquad \eta = (M_1^2 + M_2^2)^{1/2} \qquad \psi = \tan^{-1}(M_2/M_1)$$

and the magnetisation vector in each layer is $\boldsymbol{M} = \hat{x}M_1 + \hat{y}M_2$. The inhomogeneity parameter D here equals (White and Geballe 1979) $D = zS^2Jd^2/6V_0M_0^2$ where z is the number of neighbouring sites, S is the magnitude of the spin, d is the mean distance between sites, J is the interatomic exchange integral, V_0 is the unit volume and M_0 is the saturation magnetisation. Then the bifurcating solution, equation (34), describes the creation of a helicoidal magnetic structure out of a single plane impurity.

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